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Economics 6170
Appendix to Chapter 4 of Lecture Notes
On Positive Definite Quadratic Forms

1 Introduction

Let A be an $n \times n$ real symmetric matrix. It is called is *positive definite* if:

$$x'Ax > 0 \text{ for all } x \in \mathbb{R}^n, x \neq 0$$

For each $k \in \{1, \ldots, n\}$, define the submatrix $A(k)$ by:

$$A(k) = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$$

Then, the *leading principal minors* of A are:

$$\det A(1), ..., \det A(n)$$

We want to establish the following characterization result for positive definite quadratic forms (which appears in Chapter 4 of the *Lecture Notes*, but without a proof).

Theorem 1 *Suppose A is an $n \times n$ real symmetric matrix. Then A is positive definite if and only if all the leading principal minors of A are positive.*

We will deal with the two parts of the Theorem separately in the following two sections.

2 Necessity

In this section we will establish the necessity part of the theorem. That is, we will establish the following Proposition.

Proposition 1 *Suppose A is an $n \times n$ real symmetric matrix. Then A is positive definite only if all the leading principal minors of A are positive.*

Proof. We are given that A is an $n \times n$ real symmetric matrix, which is positive definite. We want to show that its leading principal minors are positive.

We break up the proof into four steps.

Step 1:

We show in this step that if B is any $m \times m$ real symmetric matrix, which is positive definite, then the determinant of B is non-zero. For if $\det B = 0$, then B is singular, and there is $x \in \mathbb{R}^m, x \neq 0$, such that:

$$Bx = 0 \tag{1}$$

Premultiplying (1) by x', we get:

$$x'Bx = x'0 = 0$$

which contradicts the fact that B is positive definite (since $x \neq 0$). This establishes that $\det B \neq 0$.

Step 2:

We show in this step that $A(k)$ is positive definite for each $k \in \{1, ..., n\}$. Given $k \in \{1, ..., n\}$, let z be an arbitrary vector in $\mathbb{R}^k$, with $z \neq 0$. Define $x = (z_1, ..., z_k, 0, ..., 0)$ in $\mathbb{R}^n$; then $x \neq 0$. Then, by performing the multiplications involved, one can verify that:

$$x'Ax = z'A(k)z \tag{2}$$

Since A is positive definite, and $x \in \mathbb{R}^n$ with $x \neq 0$. the left-hand side of (2) is positive. So, $z'A(k)z > 0$, and this establishes that $A(k)$ is positive definite.

Step 3:

In this step, we show that if B is any $m \times m$ real symmetric matrix, which is positive definite, then the determinant of B is positive. For this purpose, we define the $m \times m$ matrix $C(t)$ as follows:

$$C(t) = tB + (1-t)I \text{ for all } t \in [0, 1]$$

For each $t \in [0, 1]$, $C(t)$ is positive definite. For if $z \in \mathbb{R}^m$ with $z \neq 0$, then for each $t \in [0, 1]$,

$$z'C(t)z = tz'Bz + (1-t)z'z > 0$$

since $z'Bz > 0$ (using the fact that B is positive definite, and $z \neq 0$) and $z'z > 0$ (using the fact that $z \neq 0$). Thus, by Step 1, the determinant of $C(t)$ is non-zero for every $t \in [0, 1]$.

Now $C(0) = I$, and so $\det C(0) = 1 > 0$. Also, $C(1) = B$, and so $\det C(1) = \det B$. Thus if $\det C(1) = \det B < 0$, then there must be some $t' \in (0, 1)$, such that $\det C(t') = 0$. [We get this conclusion by noting that $\det C(t)$ is a function of t on $[0, 1]$, which is a polynomial in t, and hence continuous on $[0, 1]$, so that the Intermediate Value Theorem for continuous functions applies]. This is a contradiction, since we had concluded in the previous paragraph that $\det C(t)$ is non-zero for *every* $t \in [0, 1]$.

Thus, $\det B$ must be positive, since we have ruled out the possibility of $\det B = 0$ as well as the possibility of $\det B < 0$.

Step 4:

Using Steps 2 and 3, it follows that $\det A(k) > 0$ for each $k \in \{1, ..., m\}$. That is, all the leading principal minors of A are positive. ■

3 Sufficiency

In this section we will establish the sufficiency part of the theorem. That is, we will establish the following Proposition.

Proposition 2 *Suppose A is an $n \times n$ real symmetric matrix. If all the leading principal minors of A are positive, then A is positive definite.*

The proposition is a direct consequence of the following lemma, which looks like a stronger result but is actually equivalent in content.

Lemma 1 *Suppose A is an $n \times n$ real symmetric matrix. If all the leading principal minors of A are positive, then $A(k)$ is positive definite for each $k \in \{1, ..., n\}$.*

Proof of Lemma. We proceed to prove the result by induction. The statement is clearly true when $n = 1$.

We now make the induction hypothesis that the statement is true for $n = m$, where m is some positive integer. That is, whenever A is an $m \times m$ matrix with all its leading principal minors positive, then $A(k)$ is positive definite for all $k \in \{1, ..., m\}$.

We will now show that the statement is then true also for $n = m + 1$. [Since the statement is true for $n = 1$, this will establish that the statement is true for all $n \in \mathbb{N}$].

Let us be explicit about what is given, and what we need to prove. We are given an $(m+1) \times (m+1)$ matrix A, with all its leading principal minors positive. We have to show that $A(k)$ is positive definite for all $k \in \{1, ..., m + 1\}$. What we are given ensures that all the leading principal minors of the $m \times m$ matrix $A(m)$ are positive. By the induction hypothesis therefore we already know that $A(k)$ is positive definite for all $k \in \{1, ..., m\}$. Thus, all we need to establish is that:

$$A(m + 1) \ \text{ is positive definite}$$

Step 1:

To this end, define the $(m + 1) \times (m + 1)$ matrix B as follows:

$$B = \begin{bmatrix} a_{11} & \cdots & a_{1m} & a_{1m+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} & a_{mm+1} \\ a_{m+11} & \cdots & a_{m+1m} & b_{m+1m+1} \end{bmatrix}$$

where:

$$b_{m+1m+1} = a_{m+1m+1} - \frac{\det A(m + 1)}{\det A(m)} \tag{3}$$

Thus, B differs from $A = A(m + 1)$ in only the entry in row $(m + 1)$ and column $(m + 1)$, as described in (3). Calculating the determinant of B by expanding by the last column of B, and denoting by C_{im+1} the co-factor of $A = A(m + 1)$ corresponding to the entry a_{im+1} for $i = 1, ..., m + 1$, we get:

$$\begin{aligned} \det B &= a_{1m+1}C_{1m+1} + \cdots + a_{mm+1}C_{mm+1} + b_{m+1m+1} \det A(m) \\ &= a_{1m+1}C_{1m+1} + \cdots + a_{mm+1}C_{mm+1} + a_{m+1m+1} \det A(m) - \left[\frac{\det A(m + 1)}{\det A(m)}\right] \det A(m) \\ &= a_{1m+1}C_{1m+1} + \cdots + a_{mm+1}C_{mm+1} + a_{m+1m+1}C_{m+1m+1} - \left[\frac{\det A(m + 1)}{\det A(m)}\right] \det A(m) \\ &= \det A(m + 1) - \det A(m + 1) = 0 \end{aligned} \tag{4}$$

Thus, B is singular, and so the $(m+1)$ column vectors of B are linearly dependent. That is, there exist $\lambda_1, ..., \lambda_{m+1}$, not all equal to zero such that:

$$\lambda_1 B^1 + \cdots + \lambda_{m+1} B^{m+1} = 0 \tag{5}$$

If $\lambda_{m+1} = 0$, then (4) would imply that $B^1, ..., B^m$ are linearly dependent vectors in $\mathbb{R}^{m+1}$. This, in turn implies that $A^1(m), ..., A^m(m)$ are linearly dependent vectors in $\mathbb{R}^m$. But, that contradicts the fact that $\det A(m) \neq 0$. Thus, $\lambda_{m+1} \neq 0$, and so we can write:

$$B^{m+1} = c_1 B^1 + \cdots + c_m B^m \tag{6}$$

where $c_i = -(\lambda_i / \lambda_{m+1})$ for $i = 1, ..., m$.

Step 2:
Define the $(m+1) \times (m+1)$ matrix C as follows:

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & -c_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -c_m \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{7}$$

Then, we have:

$$BC = [B^1 \cdots B^m \; 0] \tag{8}$$

where the $(m+1)\times(m+1)$ matrix on the right hand side of (8) has B^i as its i th column for $i = 1, ..., m$, and the zero vector in $\mathbb{R}^{m+1}$ as its $(m+1)$ th column. Further, using the fact that A, and therefore B, is symmetric, we have:

$$C'BC = \begin{bmatrix} b_{11} & \cdots & b_{1m} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & \cdots & b_{mm} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1m} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \equiv D \tag{9}$$

Step 3:
Let $x \in \mathbb{R}^{m+1}$, with $x \neq 0$ be given. In order to establish that $A(m+1)$ is positive definite, we have to show that:

$$x'A(m+1)x > 0$$

Note that $\det C = 1$, so C has an inverse. Define $y = C^{-1}x$. Then, $y \in \mathbb{R}^{m+1}$ and we have:

$$Cy = x \;\text{ and so } x' = (Cy)' = y'C' \tag{10}$$

Since $x \neq 0$, it follows from (10) that $y \neq 0$.
We then have:

$$y'(C'BC)y = (y'C')B(Cy) = x'Bx \tag{11}$$

Also, denoting the vector $(y_1, ..., y_m)$ in $\mathbb{R}^m$ by z, we have by (9):

$$y'(C'BC)y = y'Dy = z'A(m)z \tag{12}$$

where D is the matrix appearing in (9). Thus, (11) and (12) imply that:

$$x'Bx = z'A(m)z \tag{13}$$

Recalling the definition of B, we can write:

$$B = A(m+1) + E \tag{14}$$

where:

$$E = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -\frac{\det A(m+1)}{\det A(m)} \end{bmatrix} \tag{15}$$

Thus, (13) can be re-written as:

$$x'A(m+1)x + x_{m+1}^2 \left[-\frac{\det A(m+1)}{\det A(m)} \right] = x'A(m+1)x + x'Ex = z'A(m)z \tag{16}$$

Step 4:

There are two cases to consider: (i) $z \neq 0$; (ii) $z = 0$. In case (i), we have $z'A(m)z > 0$, since $A(m)$ is positive definite. Consequently, by (16), we have:

$$x'A(m+1)x > -x_{m+1}^2 \left[-\frac{\det A(m+1)}{\det A(m)} \right] = x_{m+1}^2 \left[\frac{\det A(m+1)}{\det A(m)} \right] \geq 0 \tag{17}$$

the last inequality in (17) following from the facts that $x_{m+1}^2 \geq 0$, and $\det A(m) > 0$, and $\det A(m+1) > 0$. Thus $x'A(m+1)x > 0$, and $A(m+1)$ is positive definite.

In case (ii), since $y \neq 0$, we must have $y_{m+1} \neq 0$. Since $x = Cy$, we also have $x_{m+1} = y_{m+1} \neq 0$. Thus, (16) yields:

$$x'A(m+1)x = -x_{m+1}^2 \left[-\frac{\det A(m+1)}{\det A(m)} \right] = x_{m+1}^2 \left[\frac{\det A(m+1)}{\det A(m)} \right] > 0 \tag{18}$$

the strict inequality in (18) following from the facts that $x_{m+1}^2 > 0$, and $\det A(m) > 0$, and $\det A(m+1) > 0$.Thus $x'A(m+1)x > 0$, and $A(m+1)$ is positive definite.// ■